

About Energy

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1. Definition of Work

Work is defined as

$$(Eq 1) \quad W = \int_{Position_1}^{Position_2} \vec{F} \cdot d\vec{x}$$

Newton's second law of motion is

$$(Eq 2) \quad \vec{F} = \frac{d\vec{p}}{dt}$$

So we can substitute Eq 2 into Eq 1 and get

$$(Eq 3) \quad W = \int_{Position_1}^{Position_2} \frac{d\vec{p}}{dt} \cdot d\vec{x}$$

2. Classical Conservation of Energy

In classical mechanics, momentum is defined as

$$(Eq 4) \quad \vec{p} = m\vec{v}$$

Taking the derivative with respect to time of both sides of Eq 4, we get

$$(Eq 5) \quad \vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt}(m\vec{v}) = m \frac{d\vec{v}}{dt} = m\vec{a}$$

Substituting this into Eq 3, we get

$$(Eq 6) \quad W = \int_{Position_1}^{Position_2} m \frac{d\vec{v}}{dt} \cdot d\vec{x} = \int_{Position_1}^{Position_2} m d\vec{v} \cdot \frac{d\vec{x}}{dt} = \int_{Position_1}^{Position_2} m d\vec{v} \cdot \vec{v} = \int_{Position_1}^{Position_2} m \vec{v} \cdot d\vec{v}$$

$$\text{therefore} \quad W = \frac{1}{2} m v_2^2 - \frac{1}{2} m v_1^2 = T_2 - T_1$$

Eq 6 shows that the work done in getting a mass =m from position 1 to position 2 is just the difference in kinetic energy between position 2 and position 1.

However, there is another way to calculate the work done. If the underlying force on mass =m has no curl (also called "vorticity" or "rotation": think tornado), ie

$$(Eq 7) \quad \vec{\nabla} \times \vec{F} = 0$$

then there is a scalar U such that its gradient is proportional to the force; ie

$$(Eq 8) \quad \vec{F} = -\vec{\nabla} U$$

This is because of a mathematical theorem that states that the curl of any gradient is zero.

Then we can calculate the work from Eq 1 as follows:

$$(Eq 9) \quad W = \int_{Position_1}^{Position_2} \vec{F} \cdot d\vec{x} = \int_{Position_1}^{Position_2} -\vec{\nabla} U \cdot d\vec{x} = \int_{Position_1}^{Position_2} \vec{\nabla} U \cdot d\vec{x} = U_1 - U_2$$

Eq 6 and Eq 9 show two different ways to calculate work in getting mass =m from position 1 to position 2. Since both of these calculations are independent of the path that the mass takes (they depend only on starting and ending positions), we can set the two

different calculations of work equal to each other:

$$(Eq\ 10) \quad W = T_2 - T_1 = U_1 - U_2$$

Eq 10 is independent of the path. Gathering like terms together, we get

$$(Eq\ 11) \quad E = T_1 + U_1 = T_2 + U_2$$

This says that kinetic energy plus potential energy is the total energy, which is independent of the path that mass $=m$ takes. The total energy is the same for all paths and is equal to only the starting and/or ending positions. This (Eq 11) is the **law of conservation of energy**. In the derivation, we see that kinetic energy is the energy of motion and potential energy is the energy associated with the underlying force. Forces that make the law of conservation of energy true, are called **conservative forces**. They all have curl equal to zero.

Examples of conservative forces:

- Gravity
- Electromagnetic forces
- Strong nuclear force (responsible for holding the atom together)
- Weak nuclear force (responsible for nuclear decay)

Examples of nonconservative forces:

- Friction (due to Van Der Waals forces, ie electromagnetic dipoles)

In nonconservative forces, the potential energy makes no sense, because U is undefined. So the law of conservation of energy applies only to those forces which are conservative. However, in all realistic physical situations, nonconservative forces result from statistical behavior of conservative forces. So you will later find out that the total energy for a realistic physical system is

$$(Eq\ 12) \quad E = T + Q + U + V$$

where T =kinetic energy of the center of mass, Q =internal heat (statistical kinetic energy of all the particles of mass about the center of mass), U =potential energy of center of mass, and V =self-potential energy (potential energy of all the particles of mass about the center of mass). Eq 12 is the more general form of energy for the law of conservation of energy.

3. Relativistic Conservation of Energy

According to Albert Einstein's Theory of Relativity, two observers that are in different states of motion, can not agree on what events are simultaneous. The result of that discovery is that time can not be viewed as absolute. So, even though all the equations in section 1 are valid in Relativity, you have to be careful, which time you use to take the derivative with respect to.

In Relativity, "proper time" is the time which passes for events that are taken to be at relative rest. It is designated " τ ", whereas general time is designated " t ". Using this we end up with momentum being

$$(Eq\ 13) \quad \vec{p} = m \frac{d\vec{x}}{d\tau} = m \frac{d\vec{x}}{dt} \cdot \frac{dt}{d\tau} = m \vec{v} \frac{dt}{d\tau} = \gamma m \vec{v}$$

This is a sort-of definition of momentum in general time. We need one more piece of information before we can calculate work in Relativity: what is γ ? To find that, we need

to relate proper time to general time. Special Relativity prescribes the following relation

$$(Eq 14) \quad ds^2 = -c^2(d\tau)^2 + (dx')^2 + (dy')^2 + (dz')^2 = -c^2(dt)^2 + (dx)^2 + (dy)^2 + (dz)^2$$

Basically this says that the distance between any two nearby events is the same no matter who or how those events are being viewed. Eq 14 is valid in flat space-time, which basically involves events with no gravity. Since proper time is the time associated with relative rest, we can divide both sides by $d\tau$ and set the proper velocity components equal to zero. So

$$(Eq 15) \quad \left(\frac{ds}{d\tau}\right)^2 = -c^2 = \gamma^2(-c^2 + v_x^2 + v_y^2 + v_z^2) = \gamma^2(-c^2 + v^2)$$

Solving for γ , we get

$$(Eq 16) \quad \gamma = \frac{dt}{d\tau} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

for flat space-time. Now we can substitute Eq 16 into Eq 13, and that result into Eq 3, resulting in

$$(Eq 17) \quad W = \int_{Position_1}^{Position_2} \frac{d\vec{p}}{dt} \cdot d\vec{x} = \int_{Position_1}^{Position_2} \frac{d}{dt}(\gamma m \vec{v}) \cdot \vec{x} = \int_{Position_1}^{Position_2} m \vec{v} \cdot d(\gamma \vec{v})$$

Since the mass does not depend on the path or the velocity, we can take it outside the differential and integral and get

$$(Eq 18) \quad W = m \int_{Position_1}^{Position_2} \vec{v} \cdot d(\gamma \vec{v})$$

Integrating by parts, we get

$$(Eq 19) \quad W = m(\gamma_2 v_2^2 - \gamma_1 v_1^2 - \int_{Position_1}^{Position_2} \gamma \vec{v} \cdot d\vec{v}) = m(\gamma_2 v_2^2 - \gamma_1 v_1^2 - c^2 \int_{Position_1}^{Position_2} \frac{d\gamma}{\gamma^2})$$

because $d\gamma = \gamma^3 \frac{\vec{v} \cdot d\vec{v}}{c^2}$ thus $\vec{v} \cdot d\vec{v} = \frac{c^2}{\gamma^3} d\gamma$

$$W_2 = m(\gamma_2 v_2^2 + \frac{c^2}{\gamma_2}) = \gamma_2 m c^2 (\frac{v_2^2}{c^2} + \frac{1}{\gamma_2^2}) = \gamma_2 m c^2 (1 - \frac{1}{\gamma_2^2} + \frac{1}{\gamma_2^2})$$

$$W = (\gamma_2 - \gamma_1) m c^2$$

So in Special Relativity (flat space-time), the total energy is

$$(Eq 20) \quad E = \gamma m c^2 + U$$

with U being potential energy as in section 2. However, most physicists don't like to call $\gamma m c^2$ kinetic energy. The reason is that if $\gamma=1$ then $v=0$ (rest), so that the object is not moving; thus the object can't have any kinetic energy. This leads immediately to the fact that $E=mc^2$ is the rest energy of mass =m. So then, $T=(\gamma-1)mc^2$ must be the kinetic energy.

In more advanced physics studies, you will show that a Taylor expansion of $\gamma m c^2$ with v near zero will lead to the result

$$(Eq 21) \quad \gamma m c^2 = m c^2 + \frac{1}{2} m v^2 + \frac{3}{4} m \frac{v^4}{c^2} + \dots$$

Thus classical (Newtonian) concept of kinetic energy is only approximately correct to that which Relativity yields.